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POLYTROPIC SPHERES IN PALATINI $F(R)$ GRAVITY

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Abstract. We examine static spherically symmetric polytropic spheres in Palatini $f(R)$ gravity and show that no regular solutions to the field equations exist for physically relevant cases such as a monatomic isentropic gas or a degenerate electron gas, thus casting doubt on the validity of Palatini $f(R)$ gravity as an alternative to General Relativity.

1 Introduction

The quest for theories of gravity which can serve as alternatives to General Relativity (GR) has become more intense due to recent feedback from cosmology. The energy density of the universe appears to be currently dominated by a cosmological constant, or by an unknown form of energy (dark energy) that is mimicking the behaviour of a cosmological constant (Carroll 2001). The problems connected with the inclusion of such a constant in Einstein's equations have triggered research and the proposal of alternatives, one of which could be to modify GR. One of the alternative theories which has been considered is $f(R)$ gravity in the Palatini formalism. This formalism consists of considering the metric and the connection as two independent degrees of freedom and thus taking independent variations of the action with respect to each of them in order to derive the field equations, as opposed to the usual metric approach where the connection is *assumed* to be given by the Levi-Civita connection of the metric and the action is varied only with respect to the metric. When applied to the usual Einstein-Hilbert action, this procedure gives exactly Einstein's equations and the Levi-Civita formula for the connection, but when applied to a generic action, the Palatini and metric variational approaches give different results. We consider here an action given by $S = \int d^4x \sqrt{-g} f(R) / 16\pi + S_M(g_{\mu\nu}, \psi)$ (with units in which $c = G = 1$) where $R = g^{\mu\nu} R_{\mu\nu}$, g is the determinant of the metric $g_{\mu\nu}$, S_M is the matter action

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and ψ collectively denotes the matter fields. We show that for generic choices of $f(R)$ there are natural matter configurations (such as spherical systems composed of an isentropic monatomic gas or a nonrelativistic degenerate electron gas) for which *no* regular solution of the field equations can be found if one adopts the Palatini variational approach, apart from in the special case of GR, casting doubt on whether Palatini $f(R)$ gravity can be considered as a viable alternative to GR.

2 A no-go theorem for polytropic spheres: the physics behind the proof

To understand in which situations Palatini $f(R)$ gravity behaves differently from GR, let us write the field equations using just quantities built with the Levi-Civita connection of the metric (we denote these with a “tilde”, $\tilde{}$) and *not* with the *independent* connection. (As already mentioned, these expressions for the connection are in general different in Palatini $f(R)$ gravity.) One then gets (Sotiriou 2006b)

$$\begin{aligned} \tilde{G}_{\mu\nu} = & \frac{8\pi}{F} T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(R - \frac{f}{F} \right) + \frac{1}{F} \left(\tilde{\nabla}_\mu \tilde{\nabla}_\nu - g_{\mu\nu} \tilde{\square} \right) F - \\ & - \frac{3}{2} \frac{1}{F^2} \left((\tilde{\nabla}_\mu F)(\tilde{\nabla}_\nu F) - \frac{1}{2} g_{\mu\nu} (\tilde{\nabla} F)^2 \right), \end{aligned} \quad (2.1)$$

where $\tilde{\square} \equiv g^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu$, $F(R) = \partial f / \partial R$ and $T_{\mu\nu} \equiv -2(-g)^{-1/2} \delta S_M / \delta g^{\mu\nu}$ is the usual stress-energy tensor of the matter. The first three terms of this equation essentially give GR plus a cosmological constant, while deviations away from it are introduced by the terms depending on the first and second derivatives of the function $F(R)$. It is important to note that the Ricci scalar R (built with the *independent* connection) is algebraically related to the trace of the stress energy tensor, because from the trace of the field equation one gets

$$F(R)R - 2f(R) = 8\pi T, \quad (2.2)$$

which can be solved for R . In vacuum, this equation shows that R settles to a constant value R_0 , and from Eq. (2.1) one can see that Palatini $f(R)$ gravity in vacuum reduces to GR plus a cosmological constant $\Lambda = R_0/4$. In the presence of matter described by a perfect fluid with a 1-parameter equation of state (EOS) $p = p(\rho)$ (p and ρ being the pressure and the energy density of the fluid), Eq. (2.2) shows instead that R can be expressed as a function of $T = 3p - \rho$ and therefore as a function of ρ alone. As such, the first and second derivatives of F appearing in Eq. (2.1) involve first and second derivatives of ρ . It is then apparent that Palatini $f(R)$ gravity will introduce important deviations away from GR when the density changes rapidly. An obvious example of where this happens is given by neutron stars, where the density changes rapidly when going from the core to the inner crust and from the inner crust to the outer crust. In Barausse *et al.* (2007a) we have indeed studied static and spherically symmetric neutron star models in Palatini $f(R)$ gravity, and have found that the deviations away from GR can be very important for forms of $f(R)$ such as those expected from cosmology.

Another place where the derivatives of the density become large is at the surface of polytropic spheres. [We recall that a polytropic EOS has the form $p = \kappa \rho_0^\Gamma$, with ρ_0 being the rest-mass density and κ and Γ (> 1) being two constants; this can be written in the equivalent form $\rho = (p/\kappa)^{1/\Gamma} + p/(\Gamma - 1)$.] To see this, let us note that from the field equations (2.1) it follows that the stress energy tensor is conserved under covariant differentiation using the Levi-Civita connections of the metric, *i.e.* $\tilde{\nabla}_\nu T^{\mu\nu} = 0$.¹ Inserting a perfect fluid stress energy tensor and a static spherically symmetric ansatz for the metric into this equation, one gets the usual Euler equation $p' = -A'(p + \rho)/2$, where we denote radial derivatives with a “prime” and $g_{tt} \equiv \exp[A(r)]$. Using now the polytropic EOS in the Euler equation: near to the surface (defined as where $p = 0$) of a polytropic sphere, one gets $p' \propto p^{1/\Gamma}$, hence $p \propto (r_{\text{out}} - r)^{\Gamma/\Gamma-1}$ (r_{out} being the radius of the sphere) and, from the polytropic EOS, $\rho \propto (r_{\text{out}} - r)^{1/\Gamma-1}$. It is now trivial to check that the second radial derivative of the density, ρ'' , diverges for $\Gamma > 3/2$, and one therefore expects major differences between Palatini $f(R)$ gravity and GR in this case. In fact, it is possible to show (Barausse *et al.* 2007a) that the corrections coming from the terms depending on the derivatives of F in Eq. (2.1) become so important that they make the curvature invariants \tilde{R} and $\tilde{R}^{\mu\nu\sigma\lambda}\tilde{R}_{\mu\nu\sigma\lambda}$ diverge. As such, there exist *no* regular solutions to the field equations of Palatini $f(R)$ gravity for polytropic spheres with $\Gamma > 3/2$, apart from in the special case of GR (which does not have this kind of problem because only the density, and not its derivatives, enters the field equations). Physically, this means that the tidal forces diverge at the surface of such objects, although the density is exactly zero there.

It was recently argued (Kainulainen *et al.* 2007) that the singularity which we found would not cast doubt on the viability of Palatini $f(R)$ gravity because of the idealized nature of the polytropic EOS and because the lengthscale on which the tidal forces diverge due to the singularity would be shorter than the lengthscale on which the fluid approximation is valid, *i.e.* the mean free path (MFP). While it is true that the polytropic EOS may be too idealized to describe the outer layers of an astrophysical star, we note that $\Gamma = 5/3$, corresponding to an isentropic monatomic gas or a degenerate non-relativistic particle gas, falls within the range not giving a regular solution. These are perfectly physical configurations which should be describable by a viable theory of gravity without resorting to further considerations of the microphysics. Alternatively, one should accept that the theory is at best *incomplete* because of being unable to describe configurations which are well-described even by Newtonian gravity. Note that this means in particular that Palatini $f(R)$ gravity does not reproduce the Newtonian limit!

About the lengthscale on which the tidal forces diverge: in a forthcoming paper (Barausse *et al.* 2007b) we will show that while this is smaller than the MFP in the particular case considered in Kainulainen *et al.* (2007) (*i.e.* a neutron star with $f(R) = R - \mu^4/R$, where $\mu^2 \sim \Lambda$, with Λ being the cosmological constant

¹Alternatively this can be derived as in GR from the diffeomorphism invariance of the matter action, see for instance De Felice & Clarke (1990), section 6.3.

needed to explain the accelerated expansion of the universe), this is a very special situation. While $f(R) = R - \mu^4/R$ can explain cosmological data without Dark Energy, there is no first principle from which to derive this functional form, and in order to justify it one has to invoke arguments based on the series expansion of the unknown $f(R)$ coming from a consistent high energy theory. As such, there is no reason to exclude the presence, in the function $f(R)$, of terms quadratic or cubic in R , and the constraints on these terms coming from solar system tests are rather loose (Sotiriou 2006a). In Barausse *et al.* (2007b), we will show that if $f(R) = R - \mu^4/R + \varepsilon R^2$ (with ε even several orders of magnitude lower than the solar system constraint), the scale on which the tidal forces diverge is *much* larger than the MFP, even for neutron stars. [This was expected: we have already shown in Barausse *et al.* (2007a) how the effect of such a tiny ε can be important in neutron star interiors]. However, even if one cancels *by hand* all of the quadratic and cubic terms from the function $f(R)$, so that $f(R) = R - \mu^4/R$, the claim of Kainulainen *et al.* (2007) still does not apply for sufficiently diffuse systems, where the scale on which the tidal forces diverge is *much* larger than the MFP.

3 Conclusion

The problems discussed here arise due to the dependence of the metric on higher order derivatives of the matter fields, and we can expect that any theory having a representation in which the field equations include second derivatives of the metric and higher than first derivatives of the matter fields will face similar problems. The same should be expected for theories which include fields other than the metric for describing the gravitational interaction (*e.g.* scalar fields) which are algebraically related to matter rather than dynamically coupled. Indeed, one can solve the field equations for the extra field and insert the solution into the equation for the metric, inducing a dependence of the metric on higher derivatives of the matter fields. An example is a scalar-tensor theory with Brans-Dicke parameter $\omega = -3/2$, which is anyway an equivalent representation of Palatini $f(R)$ gravity (Sotiriou 2006b). As such, our results cast doubt on the viability of theories including higher order derivatives of the matter fields in one of their representations, such as generic Palatini $f(R)$ gravity or $\omega = -3/2$ scalar-tensor theory.

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